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THE OPTIMAL DESIGN OF BEAM-COLUMNS

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Abstract—We solve the problem of minimizing the midspan transverse deflection of a simply supported elastic beam-column of given mass, i.e. of an elastic bar acted upon simultaneously by an axial compression and a transverse concentrated load. Solutions are presented for bars whose cross section is of sandwich or solid construction. It is shown that optimization leads to a substantial decrease in the transverse deflection. Simple formulas are suggested relating the minimum midspan transverse deflection to the applied axial compression and transverse concentrated load. Alternatively, these formulas could be used to calculate the maximum allowable axial compression (transverse load) for prescribed midspan deflection and transverse load (axial compression). Finally, simple approximate analytical expressions are given for the optimal design.

INTRODUCTION

Optimal design of structural members (ties, beams, columns, plates, etc.) that perform a given function with the least amount of material has received considerable attention over the years (see, e.g. the review papers [1, 2]). Thus, e.g. the member may be required to act as a beam for a part optimal design of structural members that have to perform several functions at different times during their design life [3–8]. Thus, e.g. the member may be required to act as a beam for a part of its design life and as a column for the rest, but not both at any given time.

On the other hand, it is not an uncommon situation is structural design when a member is acted upon simultaneously by an axial compression and a transverse load. Such members are usually called beam-columns. The aim of the present paper is to design the beam-columns in such a way as to minimize the resulting maximum deflection. Specifically, we solve the problem of a simply supported elastic beam-column of given mass (volume) under the simultaneous action of an axial compression and a transverse concentrated load. With little modification, this solution is also valid for a cantilevered beam-column.

Solutions are presented for beam-columns whose cross section is of sandwich or solid construction. It is shown that optimization leads to a very substantial decrease in the transverse deflection. Moreover, simple formulas are suggested relating the minimum midspan deflection to the applied axial compression and transverse concentrated load. Alternatively, these formulas could be used to calculate the maximum allowable axial compression, if the maximum midspan deflection and the transverse load are prescribed. Simple analytical expressions are given for the optimal design.

1. PROBLEM STATEMENT AND OPTIMALITY CONDITION

The following optimization problem is posed. It is required to minimize the maximum deflection $y^*(l/2)$ at the midspan of a simply supported elastic beam-column under the simultaneous action of an axial compressive force P^* and a midspan transverse concentrated force $2Q^*$. The beam-column has to achieve this goal by a suitable distribution of the available material volume, V, along its length, l.

Mathematically, the transverse deflection of the beam-column must satisfy the following differential equation and boundary conditions written in a non-dimensional form

$$\alpha^n y_{xx} + Py + Qx = 0, \qquad 0 \leq x \leq 1/2, \qquad y(0) = y_x(1/2) = 0 \tag{1}$$

where $y = y^*/l$ and subscript x denotes differentiation with respect to the dimensionless linear coordinate. Dimensionless axial compression P and one half the midspan concentrated force,

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Q, are defined by $P = P^* l^{2+n}/EcV^n$ and $Q = Q^* l^{2+n}/EcV^n$, E being the Young's modulus of the material. It is assumed that the deflection function is symmetric about the midspan, as are the second moment of area, I, and the cross-sectional area, A. The latter two are assumed to be related through $I = cA^n$, where c and n are constants defined by the cross-sectional shape. Thus, $n = 1$ represents a cross section of sandwich construction and $n = 2$ and 3- solid construction. Finally, the dimensionless area of cross section $\alpha(x)$ is defined by $\alpha(x) = Al/V$.

In dimensionless variables the condition of given material volume (isoperimetric condition) takes the following form

$$2\int_0^{1/2} \alpha(x)\,\mathrm{d}x = 1. \tag{2}$$

The optimization problem under consideration consists in determining the variation of $\alpha(x)$ along the length of the beam-column that satisfies the differential eqn (1) and isoperimetric condition (2) and minimizes its maximum transverse deflection (at midspan)

$$y(1/2) = \frac{1}{Q}\int_0^{1/2} \alpha^n y_{xx}^2\,\mathrm{d}x - \frac{P}{Q}\int_0^{1/2} y_x^2\,\mathrm{d}x \to \min. \tag{3}$$

It is easily shown that the solution of the optimization problem posed above will also be the solution of the following optimization problem. Given the isoperimetric condition (2), the allowable midspan deflection $y(1/2) = y_0$ under a transverse concentrated force $2Q$, determine the variation of $\alpha(x)$ that maximizes the permissible axial compression

$$P = \frac{\int_0^{1/2} \alpha^n y_{xx}^2\,\mathrm{d}x - Qy_0}{\int_0^{1/2} y_x^2\,\mathrm{d}x} \to \max. \tag{4}$$

The necessary optimality condition for the above-stated optimization problem is obtained by setting to zero the first variation of an auxiliary functional that includes the isoperimetric condition (2) through a Lagrange multiplier. For both statements of the problem, the necessary optimality condition is a special case of the well-known general condition [7]

$$\alpha^{n-1} y_{xx}^2 = \nu, \tag{5}$$

where the unknown constant ν is specified by the isoperimetric condition (2).

2. METHOD OF SOLUTION

In view of the boundary condition at $x = 0$ and the fact that the bending moment $(\alpha^n y_{xx})$ vanishes at the simply supported ends of the beam column, it is clear that $\alpha(0) = 0$. Hence it follows from the optimality condition (5) that y_{xx} could suffer a singularity at $x = 0$. To investigate such a possibility, let $y = B_1 x + B_2 x^m$ near $x = 0$, where m is the lowest non-integer power. Substituting for y, α and y_{xx} in the differential eqn (1), it can be shown that $m = (n+3)/(n+1)$, whence it follows that y_{xx} varies as $x^{(1-n)/(1+n)}$ near $x = 0$. In other words, y_{xx} is singular at $x = 0$ for $n > 1$. In view of the non-singular behaviour of y_{xx} near $x = 0$ and the disappearance of α from the optimality condition (5), the special case when the area of cross section and the stiffness are linearly related ($n = 1$) will be separately treated.

2.1 *Sandwich cross section* (n = 1)

The special problem of sandwich cross section is easily solved in a closed form. In this case the area function $\alpha(x)$ disappears from the optimality condition (5), whence by double integration and the application of boundary conditions, we get

$$y = -\frac{x}{2}(1-x)\sqrt{\nu}. \tag{6}$$

Substituting (6) and its derivatives into the differential eqn (1) and invoking the isoperimetric condition (2), we get the following solution

$$\alpha(x) = \frac{Px(1-x)}{2} + \frac{(12-P)x}{3}, \quad 0 \leq x \leq 1/2 \tag{7}$$

$$y_{min}(1/2) = \frac{3}{8}\frac{Q}{(12-P)}. \tag{8}$$

It is worth mentioning that the two terms in the expression for $\alpha(x)$ are the column and beam contributions, respectively, although Q does not explicitly appear in this expression. Thus, when $P = 0$, the problem reduces to that of minimizing y (1/2) under a concentrated midspan force $2Q$. In this case, the solution is known[6, 9] and coincides with that given by (7)–(8). On the other hand, when $Q = 0$, the problem reduces to that of maximizing the Euler buckling load of a column subject to the isoperimetric condition (2). The solution of this problem is also known[6, 10] and is $P_{max} = 12$. Note that in this eigenvalue problem $y(1/2)$ remains undetermined. This also follows from (8).

As mentioned in introduction, the solution (8) can also be interpreted in the following way. Given a permissible midspan deflection $y(1/2) = y_0$ under a concentrated force $2Q$, it specifies the maximum axial compression that can be applied to the beam-column without exceeding y_0

$$P_{max} = 12 - 0.375\ Q/y_0. \tag{9}$$

The effectiveness of the optimal design will be judged against a prismatic beam-column. However, before making this comparison we present an iterative scheme for the solution of the problem under consideration, when $n = 2$ or 3.

2.2 *Solid cross section* (n = 2 *or* 3)

Unlike the special case of $n = 1$, the optimization problem does not seem to have a closed form solution. A simple iterative scheme, involving only regular functions, was used to arrive at the solution. The iterative procedure involved the following sequence of steps:

(i) For given P and Q a regular function $g_i(x) = 1$ $(0 \leq x \leq 1/2)$ in the first iteration $(i = 1)$ was assumed.

(ii) $(y_x)_i = -\int_x^{1/2} \eta^{(1-n)/(1+n)} g_i(\eta)\, d\eta$.

(iii) $y_i = \int_0^x y_\eta(\eta)\, d\eta$.

(iv) $\nu_i = \left\{0.5 \Big/ \int_0^{1/2} (Py_i + Qx)^{2/(n+1)}\, dx\right\}^{n+1}$.

(v) $\alpha_i = \nu_i^{1/(n+1)}(Py_i + Qx)^{2/(n+1)}$.

(vi) $g_i(x) = -1/\{\nu_i^{n/(n+1)}(Py_ix + Q)^{(n-1)/(n+1)}\}$.
Note $g_i(0) \neq 0$.

(vii) Repeat steps (ii)–(vi), if $|\nu_{i+1} - \nu_i| > 10^{-5}$.

Instead of presenting the customary plot of the variation in $\alpha(x)$ for various values of P and n, we present below approximate analytical expressions for $\alpha(x)$ for both $n = 2$ and $n = 3$. These analytical expressions give results that differ from the "exact" numerical results by no more than 1.5%.

The choice of the analytical expression for $\alpha(x)$ when $n = 2$ or 3 was dictated by the form of the corresponding expression for $n = 1$ (7) and the following known or observed properties

(i) $\alpha(0) = 0$.

(ii) $\alpha(x)$ varies as $x^{2/(n+1)}$ in the vicinity of $x = 0$.

(iii) $\alpha(x) = (n + 3/n + 1)(2x)^{2/(n+1)}$, when $P = 0$ (see, e.g. [9]).

(iv) $\alpha(x)$ is not explicitly dependent on Q, but depends only on P.

Bearing the above considerations in mind, the following approximate analytical expression for $\alpha(x)$ was assumed

$$\alpha(x) = \frac{Px^{2/(n+1)}(1 - Ax^{\lambda})}{B} + \frac{(P_{opt} - P)}{C} x^{2/(n+1)}; \qquad 0 \leq x \leq 1/2. \tag{10}$$

Here P_{opt} refers to the Euler buckling load of the optimally designed column. It takes the values 12.00, 13.16 and 13.88 for $n = 1$, 2 and 3, respectively[6, 8]. The constant C is immediately found from the known solution for the optimal beam, because the second term in the expression for $\alpha(x)$ is the beam contribution, i.e. when $P = 0$. In fact, it is easily shown that C takes the values 3.0000, 4.9741 and 6.5432 for $n = 1$, 2 and 3, respectively. The remaining two unknown coefficients A and B and the exponent λ were determined from the following conditions resulting either from the problem itself or from an examination of the numerical solutions for $n = 2$ and 3:

(i) $\alpha(x_0) = \text{Const.}$ for all values of $P \geq 0$, where x_0 takes the values 0.309 and 0.296 for $n = 2$ and 3, respectively. This followed from an examination of the numerical solutions for $n = 2$ and 3. It is worth mentioning that a similar situation arises in the sandwich case. Indeed, it is easy to deduce from expression (7), that $x_0 = 1/3$.

(*ii*) The isoperimetric condition (2).

(iii) $\alpha_x(1/2) = 0$, when $P = P_{opt}$, where subscript x denotes differentiation with respect to this dimensionless longitudinal coordinate. This condition expresses mathematically the fact that the optimal column design is flat at midspan.

Conditions (i) and (iii) gave expressions relating the coefficients A and B to the exponent λ. Finally, condition (ii) led to a non-linear equation in λ, whose solution was found by trial and error.

The following simple analytical expressions for $\alpha(x)$ gave results that differed from the "exact" numerical results by less than 1.5%:

$$\alpha(x) = \begin{cases} \dfrac{Px^{2/3}(1 - 0.793x^{7/8})}{3.55} - \dfrac{(13.16 - P)}{4.9741} x^{2/3}; & n = 2, \\[2ex] \dfrac{Px^{1/2}\left(1 - \frac{2}{3}x^{7/8}\right)}{5.02} - \dfrac{(13.88 - P)}{6.5432} x^{1/2}; & n = 3. \end{cases} \tag{11}$$

It is worth reemphasizing that the optimal design for $n = 2$ and 3, like the design for $n = 1$ (expression 7) is independent of Q. Likewise, numerical results showed that the minimum midspan deflection is a linear function of Q.

The values of Q/y_{min} (1/2) for various values of axial compression P are tabulated below. For completeness of presentation the corresponding values for $n = 1$ are also included.

Table 1.

	$Q/y_{min}(1/2)$ for various n			Prismatic beam-column	$(1 - y_{min}/y_{prism}) \cdot 100$ where applicable		
P	1	2	3	$Q/y_{prism}(1/2)$	1	2	3
0	32.00	37.04	40.05	24.00	25.00	25.21	40.07
4	21.33	25.66	28.60	14.35	32.72	44.07	49.82
5	18.67	22.83	25.66	11.92	36.15	47.78	53.82
6	16.00	20.01	22.73	9.49	40.69	52.58	58.25
7	13.33	17.20	19.81	7.05	47.11	59.02	64.43
8	10.67	14.39	16.91	4.60	56.89	68.05	72.79
9	8.00	11.59	14.01	2.14	73.25	81.51	84.71
10	5.33	8.80	11.12	—	—	—	—
11	2.67	6.01	8.25	—	—	—	—
12	—	3.22	5.35	—	—	—	—

In order to judge the effectiveness of the optimal designs the midspan deflection of the optimal beam-column was compared with that of a prismatic beam-column of the same volume and subjected to the same axial and transverse forces, as the optimal beam-column.

The cross-sectional area of such a prismatic beam-column is $\alpha(x) = 1$, $0 \leq x \leq 1/2$ for all n, and its midspan deflection is given by [11]

$$y(1/2) = \frac{Q}{2P}\left(\frac{\tan \sqrt{(P)}/2}{\sqrt{(P)}/2} - 1\right). \tag{12}$$

Clearly, the comparison is meaningful only when P is less than the Euler buckling load of the prismatic column ($P < \pi^2$). This is true for all n.

From the tabulated values it is clear that the optimal design leads to a very substantial reduction in the transverse deflection in comparison with that of a prismatic beam-column.

As in the special case $n = 1$, the value of P for the optimal column is known to be 13.16 and 13.88, for $n = 2$ and 3, respectively [6, 8]. With this in mind, it was possible to deduce from the numerical results (to within an accuracy of $\pm 0.5\%$) the following explicit expressions for $y_{min}(1/2)$ as a function of prescribed P and Q

$$y_{min}(1/2) = \begin{matrix} 0.358\, Q/(13.16 - P), & n = 2 \\ 0.347\, Q/(13.88 - P), & n = 3. \end{matrix} \tag{13}$$

Again, an alternate interpretation of (13) is also valid: Given a permissible midspan deflection y_0 under a concentrated force $2Q$, (13) specifies the maximum axial compression that can be applied to the beam-column without exceeding y_0

$$P_{max} = \begin{matrix} 13.16 - 0.358\, Q/y_0; & n = 2 \\ 13.88 - 0.347\, Q/y_0; & n = 3. \end{matrix} \tag{14}$$

The solution of the optimization problem posed here has clearly demonstrated that optimization can lead not only to a substantial reduction in the transverse deflection, but also provide a possible design in situations where a prismatic beam-column could not even be contemplated. However, the limitations of the linear elastic theory used here should be borne in mind in judging the results when $y_{min}(1/2)$ is very large (Q/y_{min} is small).

Finally, it should be pointed out that with minor reinterpretation of P and Q, the solution presented here is also valid for a cantilevered beam-column.

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